Non-regular languages

(Pumping Lemma)

- Introduction

Non Regular language

How can we prove that a language is not regular?

$$\{a^n b^n : \quad n \geq 0\}$$

Non-regular languages

$$\{vv^R : \quad v \in \{a,b\}^*\}$$

Regular languages

$$a^*b$$

$$b^*c + a$$

$$b + c(a+b)^*$$

etc...

How can we prove that a language L is not regular?

Prove that there is no DFA or NFA or RE that accepts L

Difficulty: this is not easy to prove

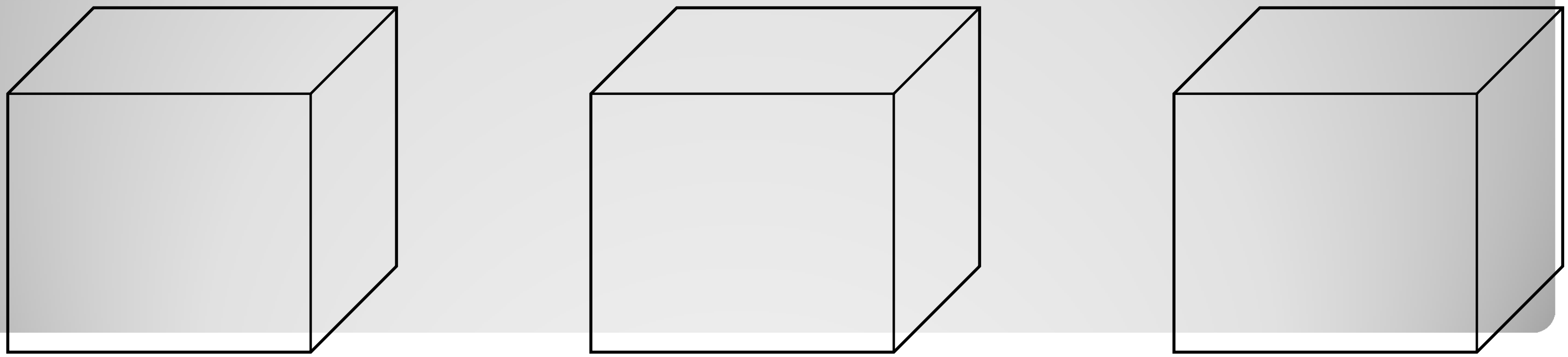
(since there is an infinite number of them)

Solution: use the Pumping Lemma !!!

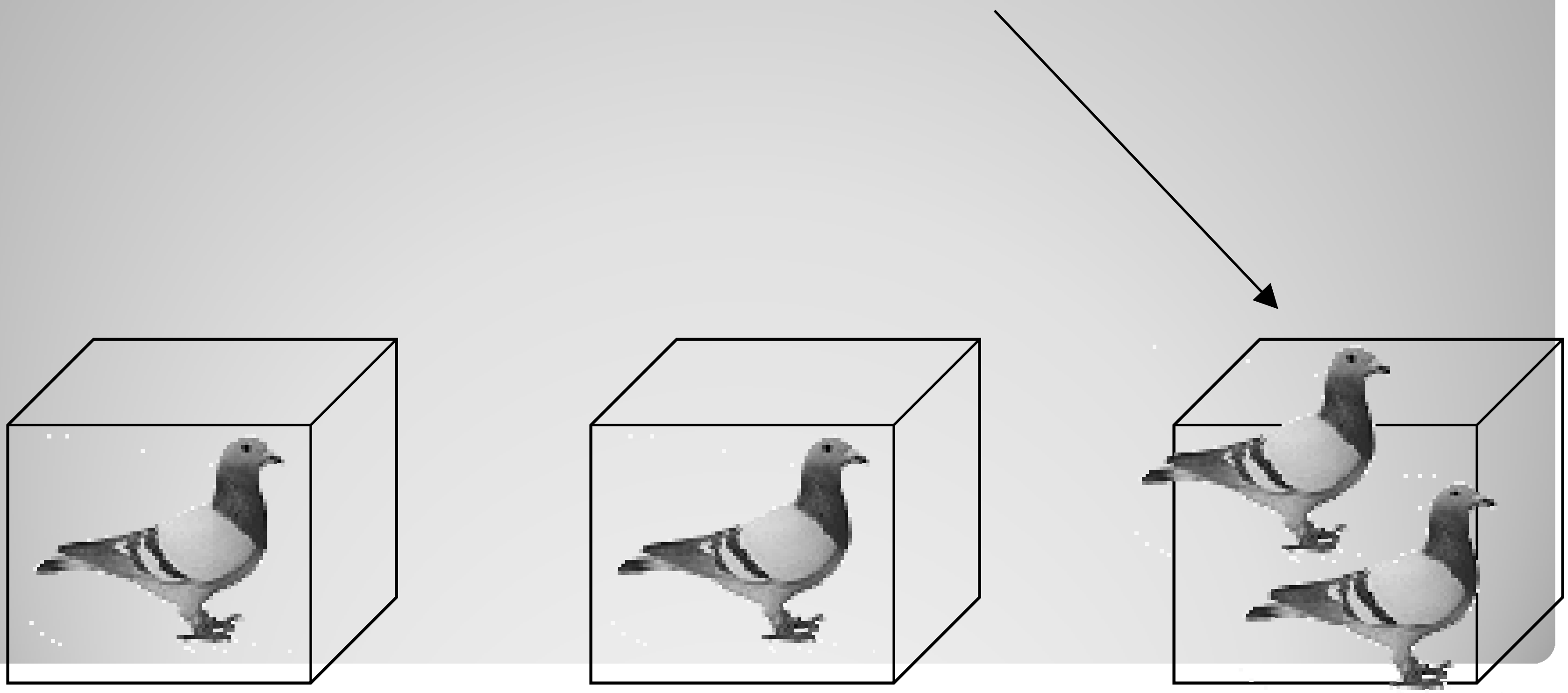
The Pigeonhole Principle

4 pigeons

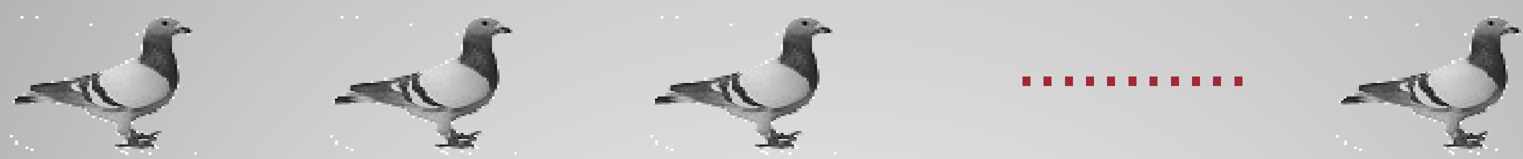
3 pigeonholes



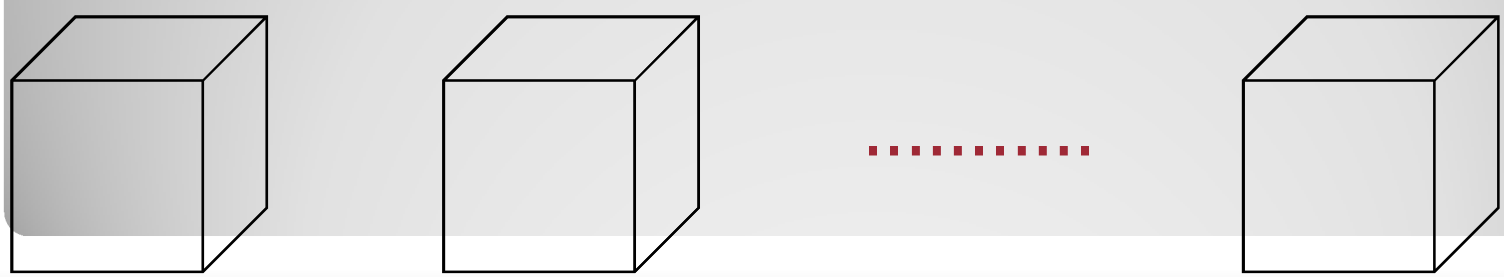
A pigeonhole must contain at least two pigeons



n pigeons



m pigeonholes $n > m$



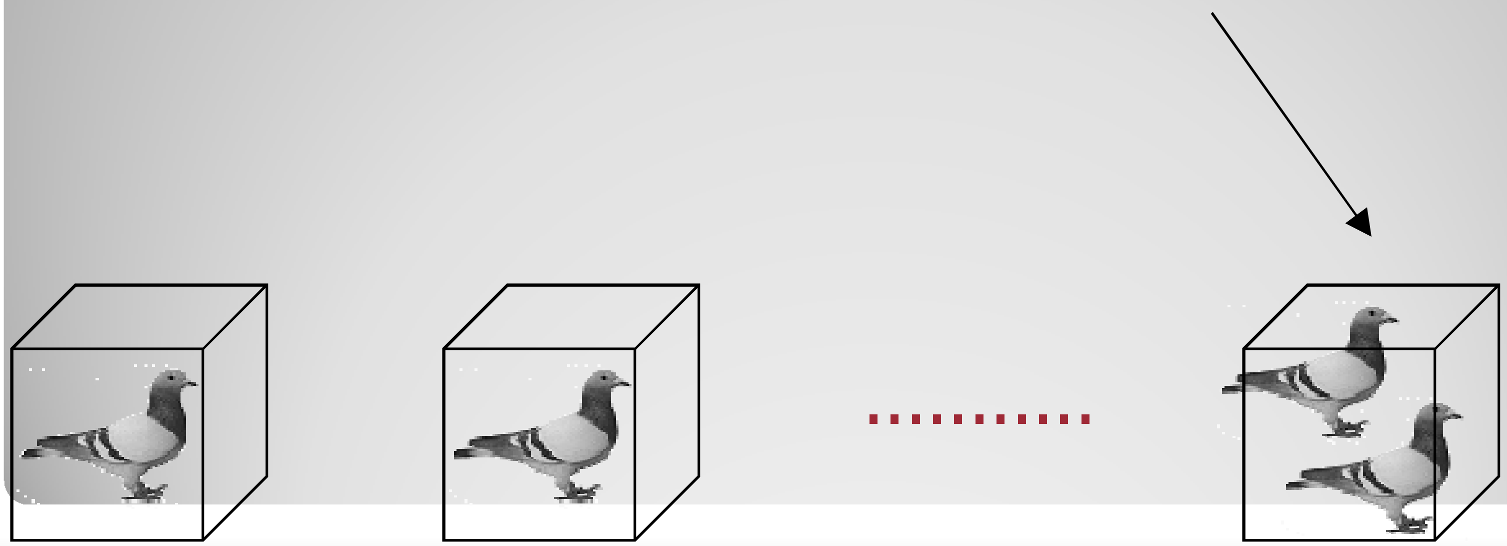
The Pigeonhole Principle

- n pigeons

m pigeonholes

$$n > m$$

There is a pigeonhole with at least 2 pigeons

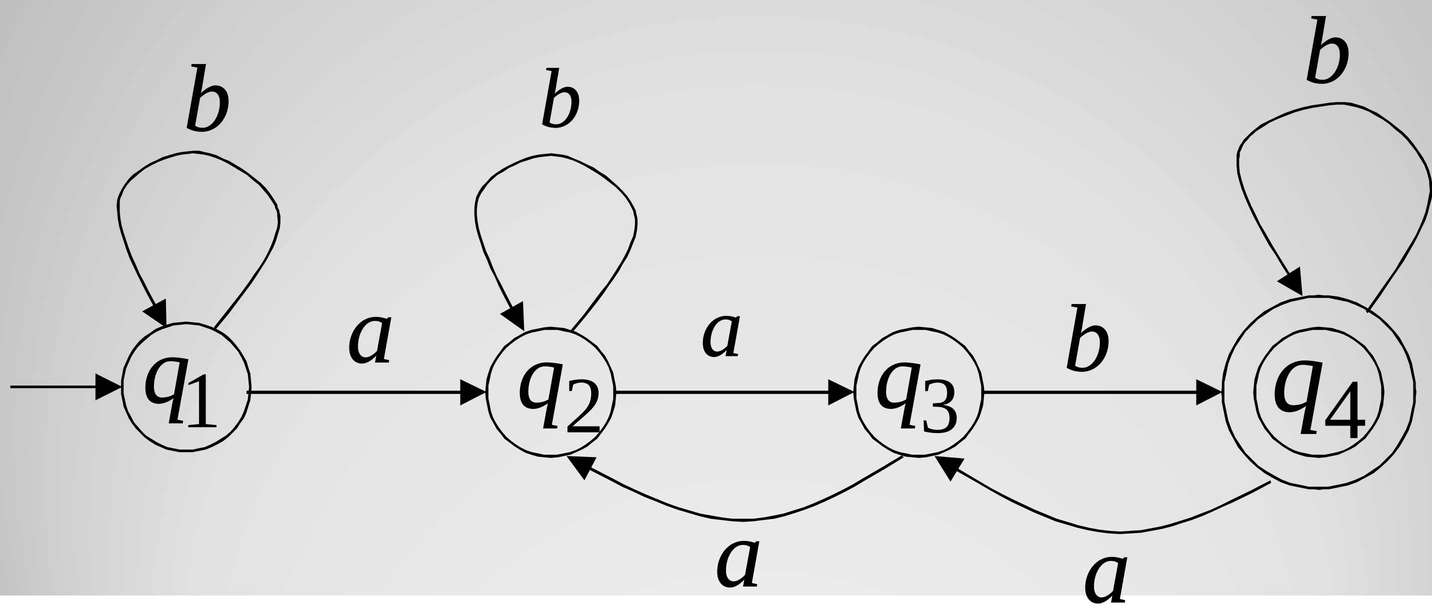


The Pigeonhole Principle

and

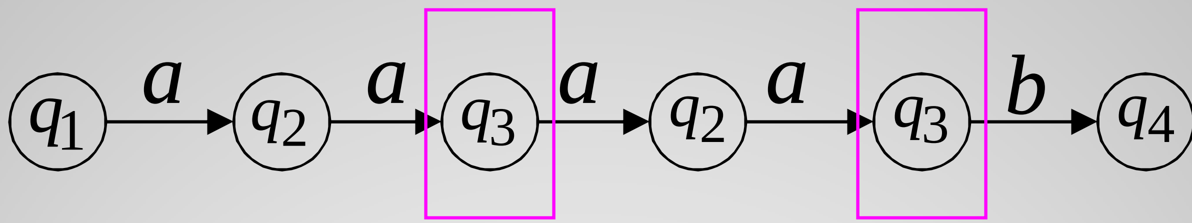
DFAs

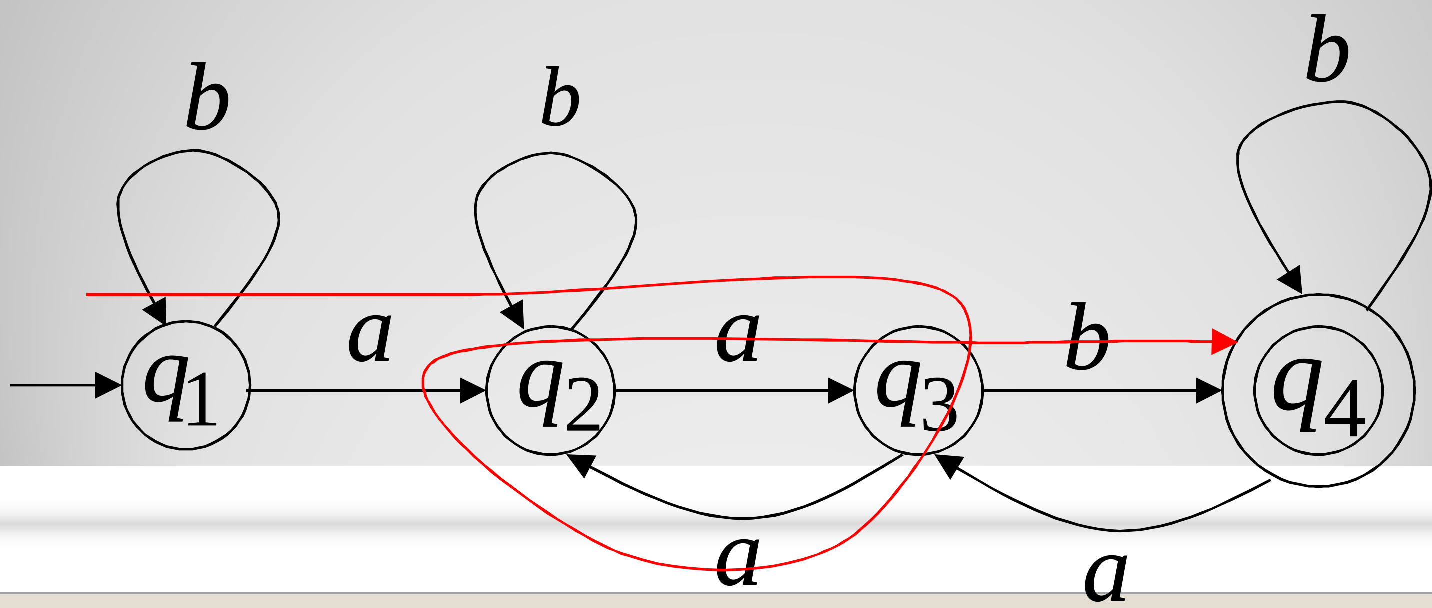
Consider a DFA with 4 states



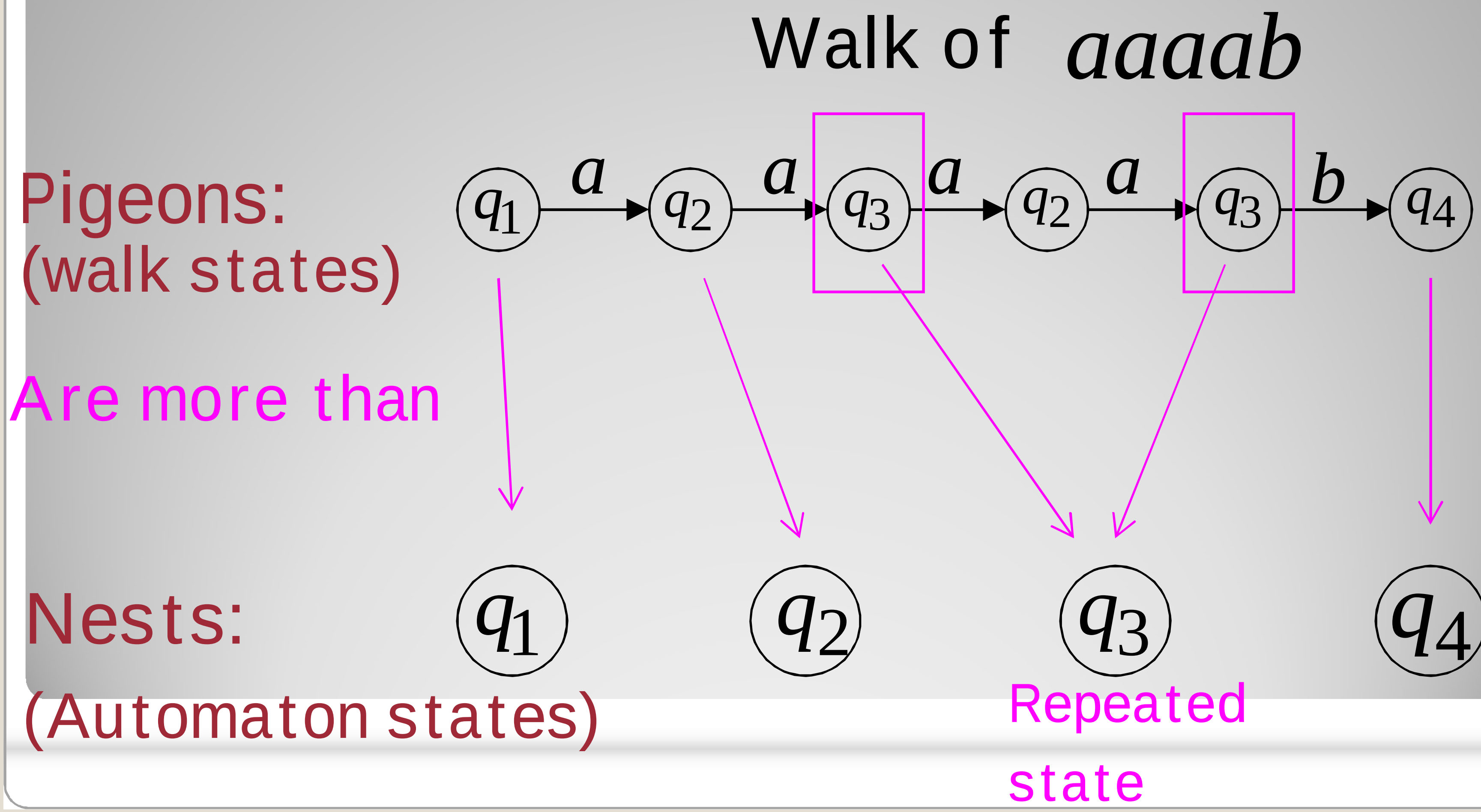
Consider the walk of a "long" string: $aaaab$
(length at least 4)

A state is repeated in the walk of $aaaab$





The state is repeated as a result of the pigeonhole principle

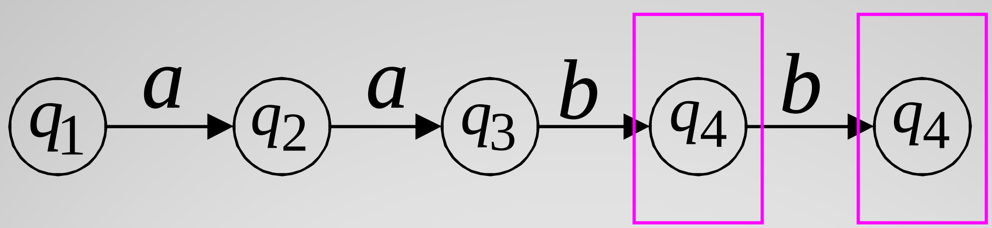


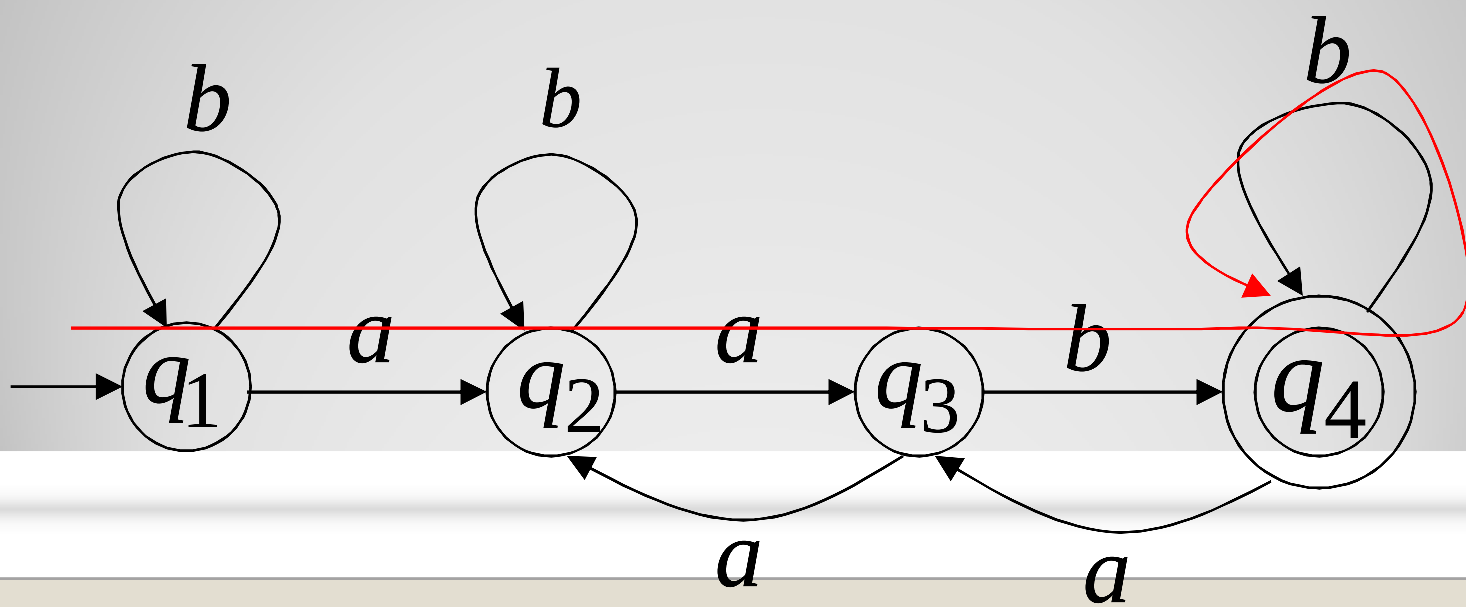
Consider the walk of a "long" string: *aabb*

(length at least 4)

Due to the pigeonhole principle:

A state is repeated in the walk of *aabb*



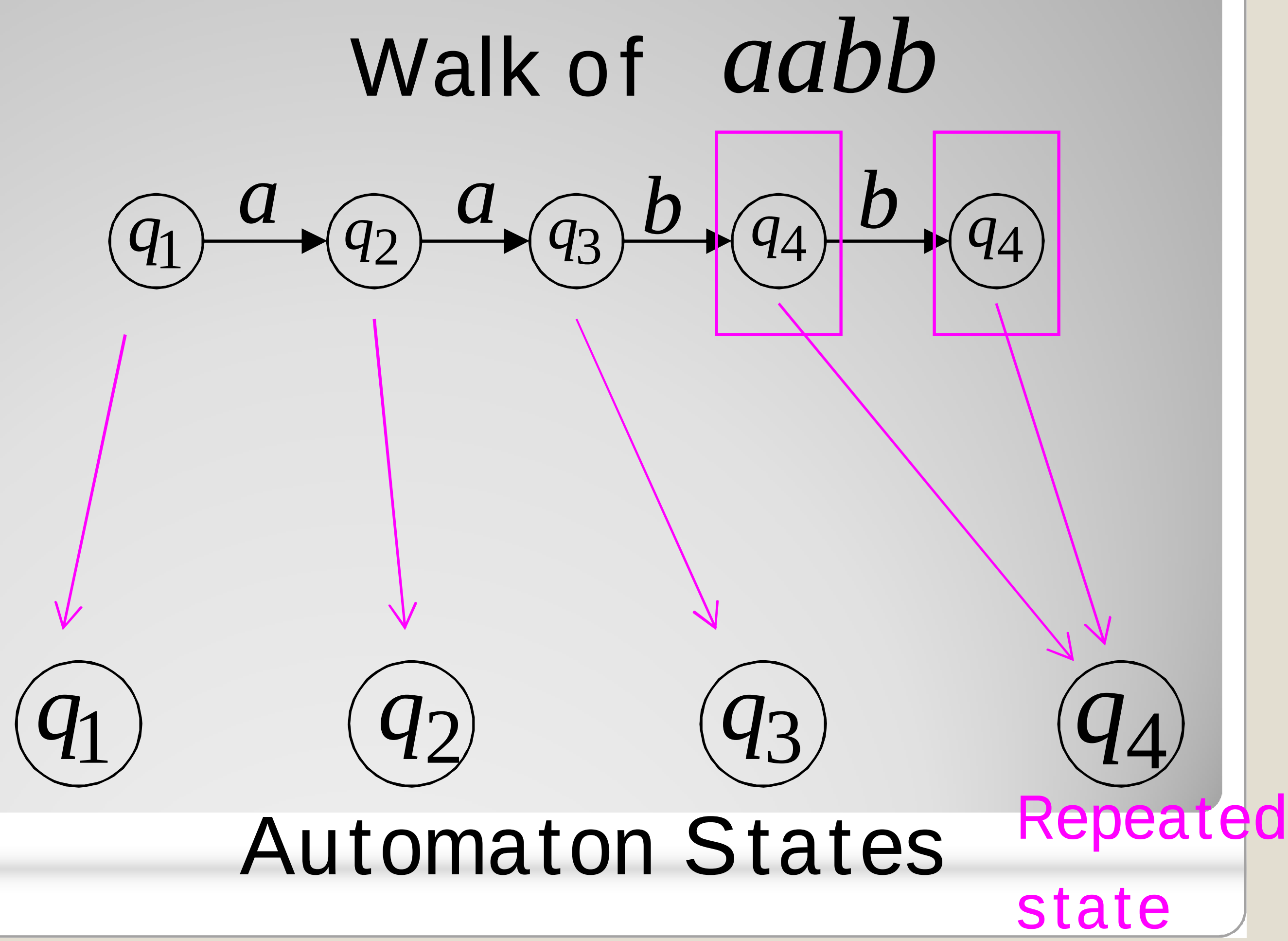


The state is repeated as a result of the pigeonhole principle

Pigeons:
(walk states)

Are more than

Nests:
(Automaton states)

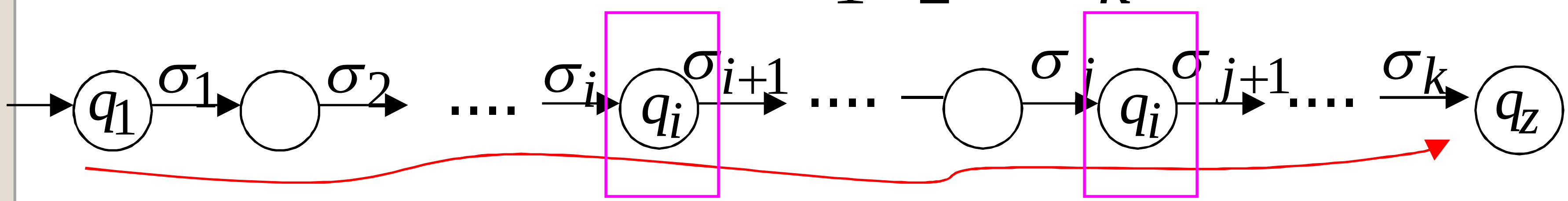


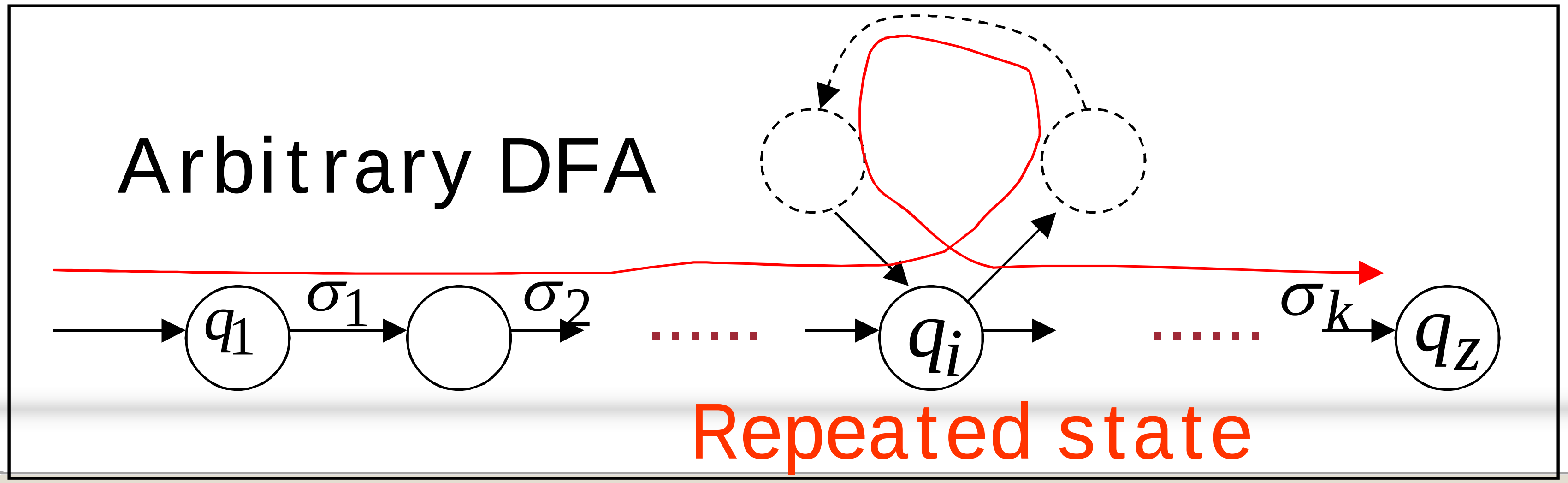
In General: If $|w| \geq$ #states of DFA,

by the pigeonhole principle,

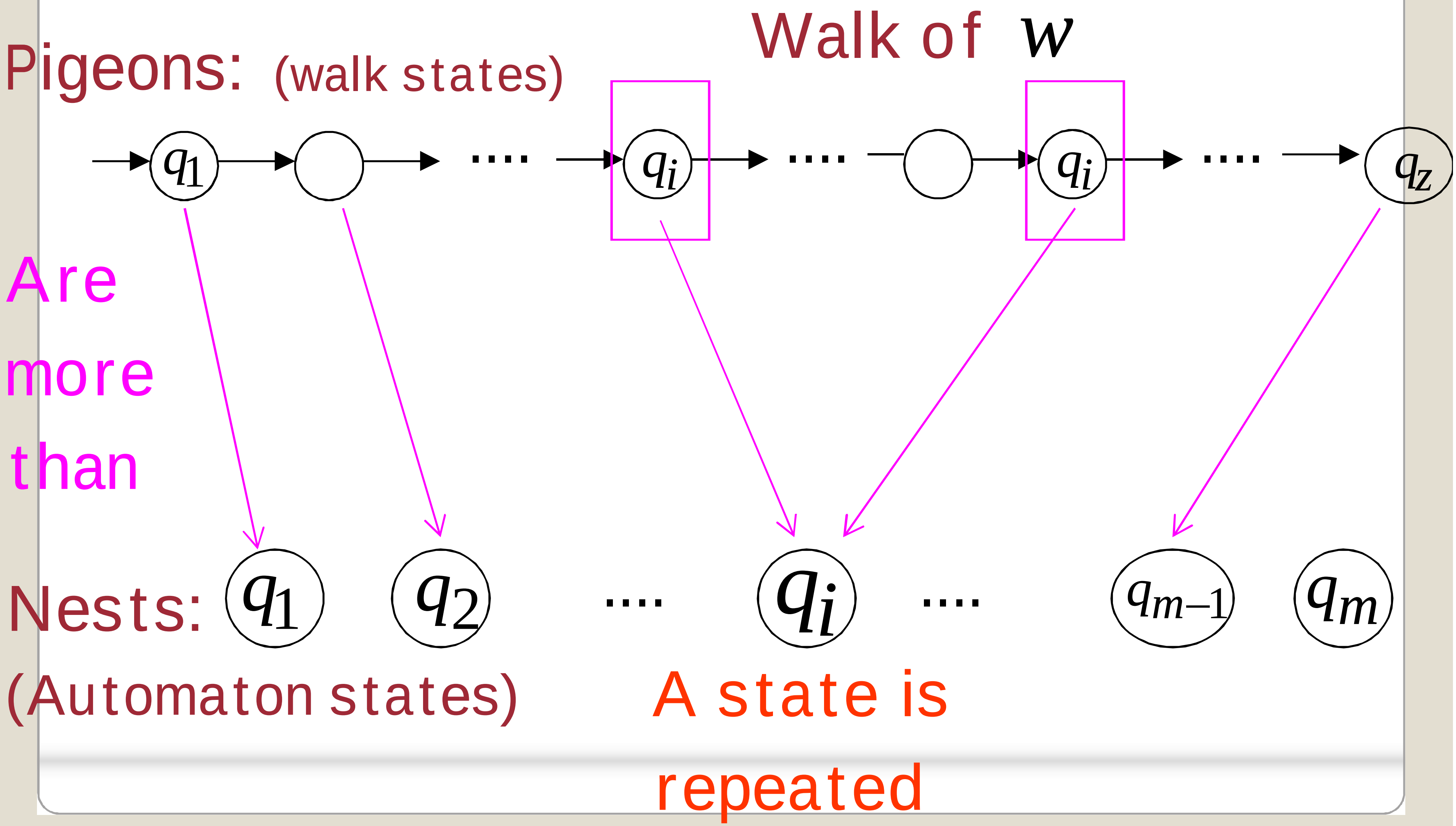
a state is repeated in the walk w

Walk of $w = \sigma_1 \sigma_2 \cdots \sigma_k$





$$|w| \geq \#\text{states of DFA} = m$$

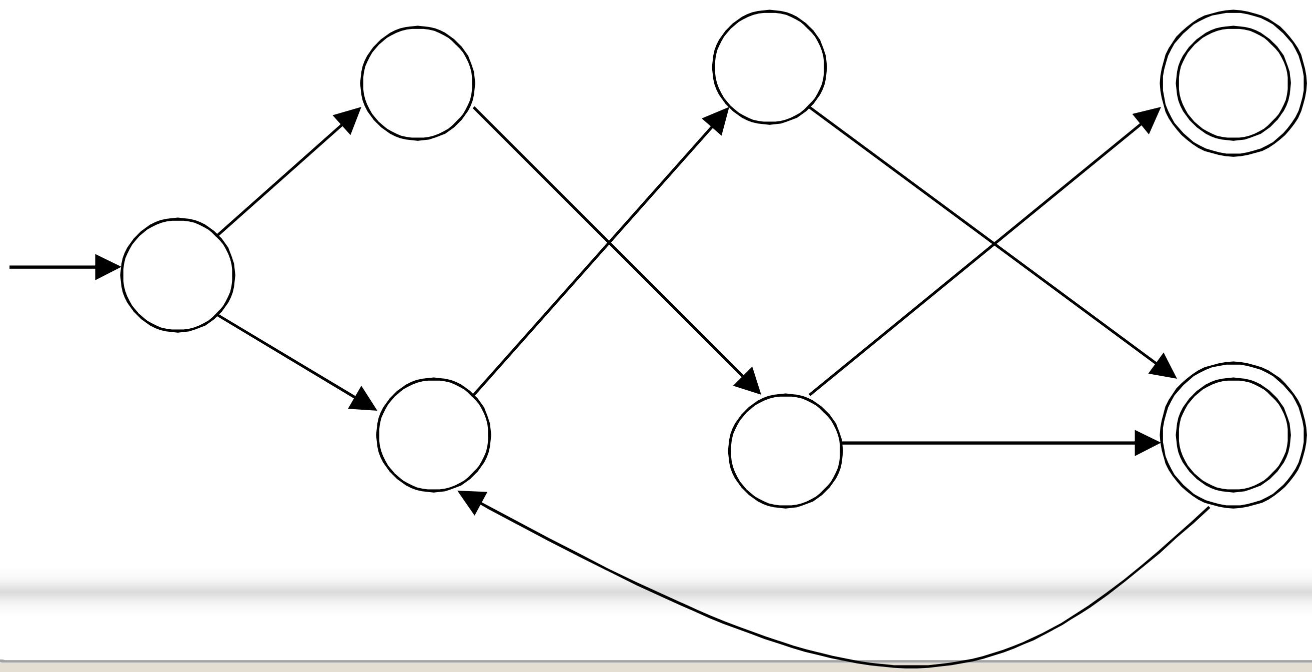


The Pumping Lemma

Take an infinite regular language L

(contains an infinite number of strings)

There exists a DFA that accepts L



m

states

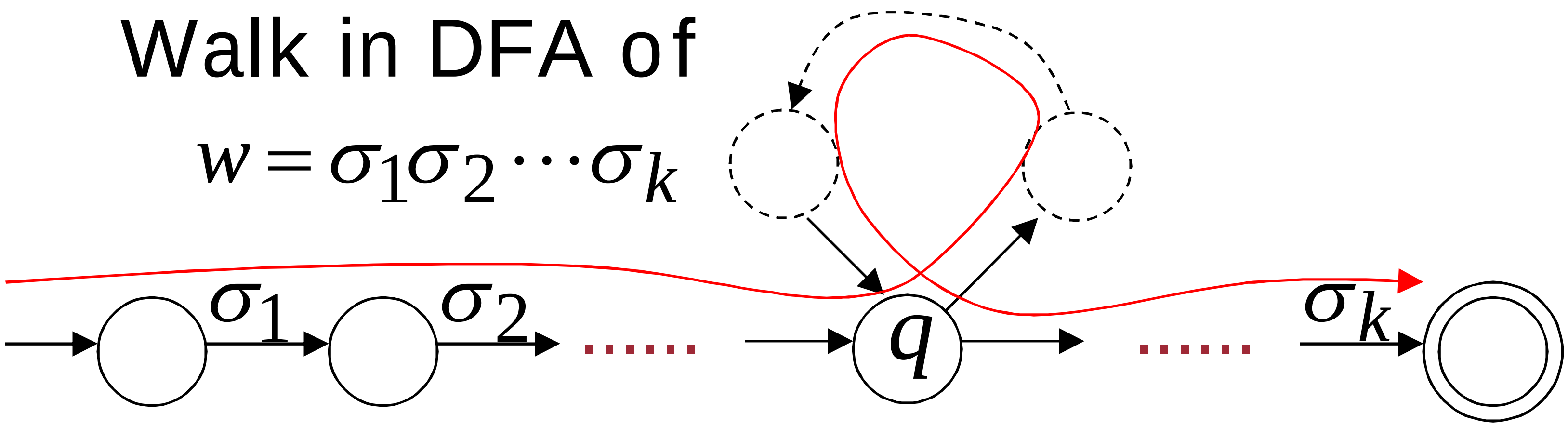
Take string $w \in L$ with $|w| \geq m$

(number of states of DFA)

then, at least one state is repeated in the walk of w

Walk in DFA of

$w = \sigma_1 \sigma_2 \cdots \sigma_k$

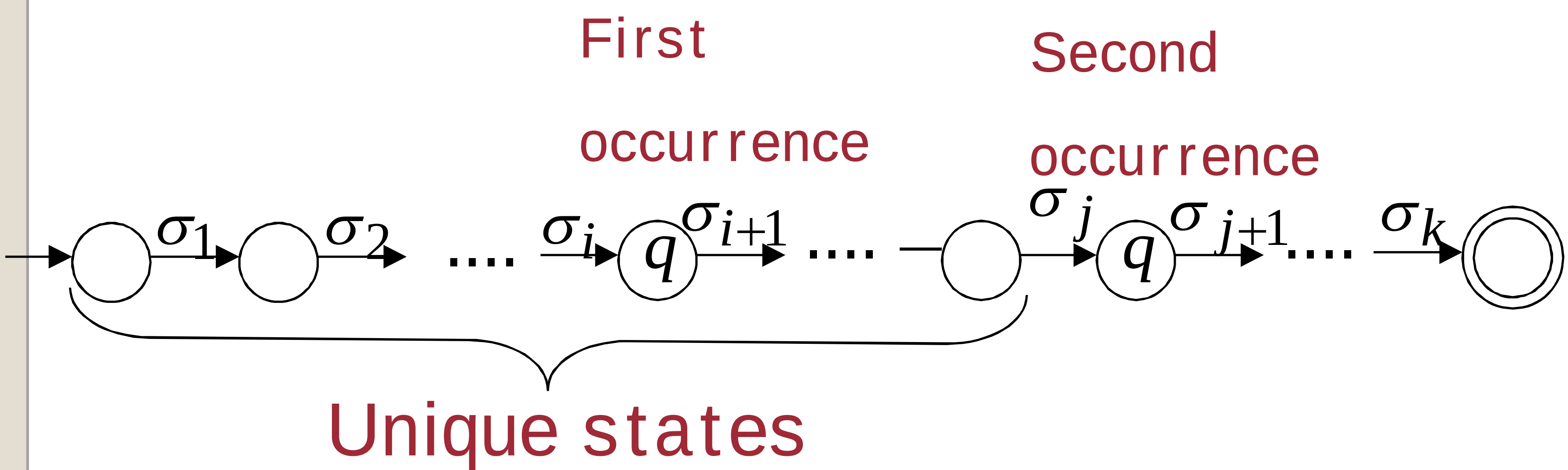


Repeated state in DFA

There could be many states repeated

Take q to be the first state repeated

One dimensional projection of walk w:

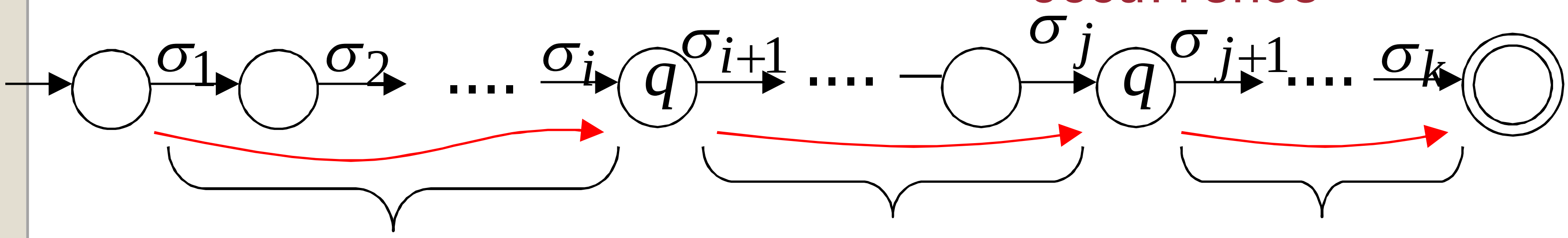


We can write $w = xyz$

One dimensional projection of walk w:

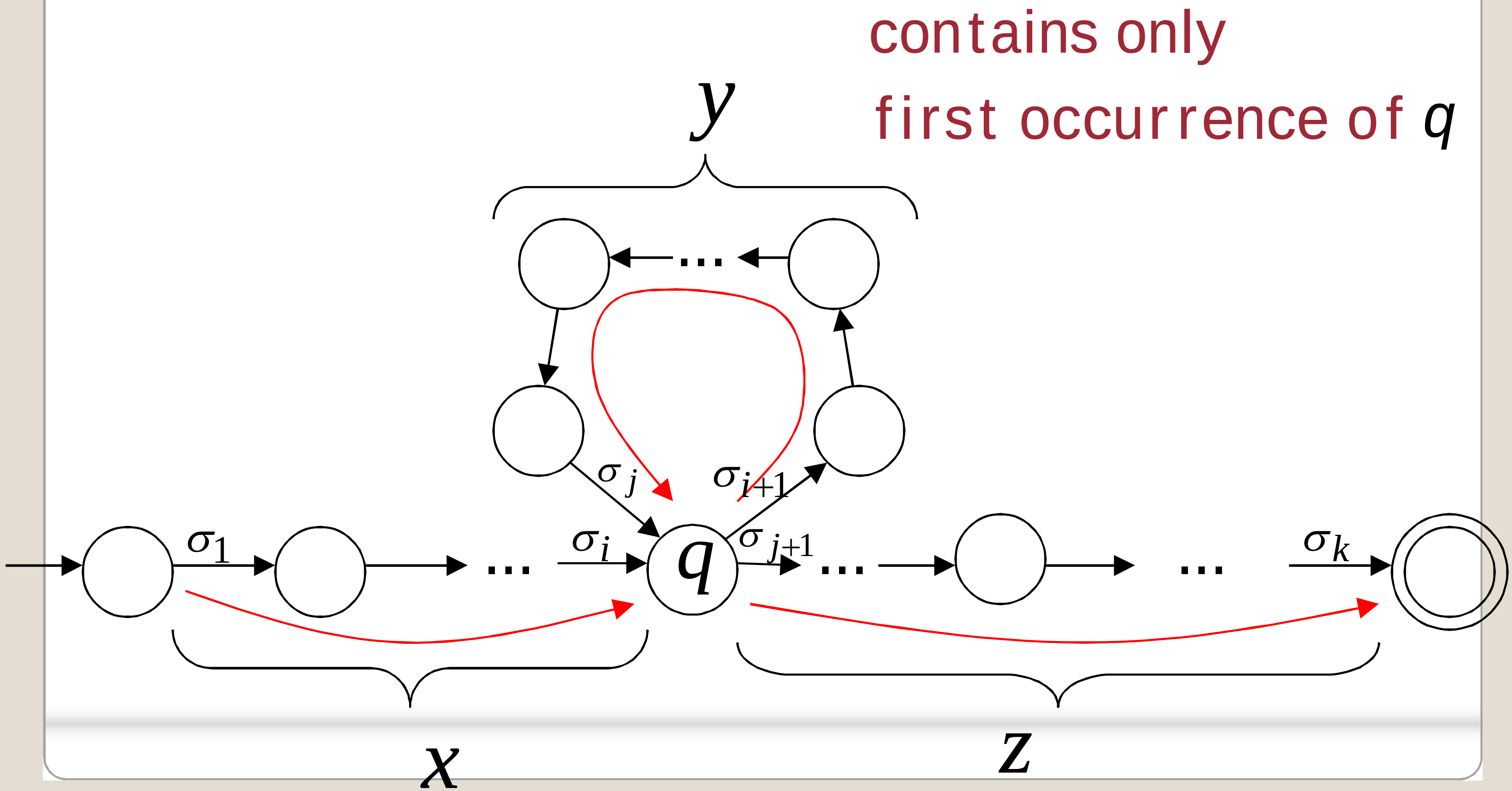
First occurrence

Second occurrence

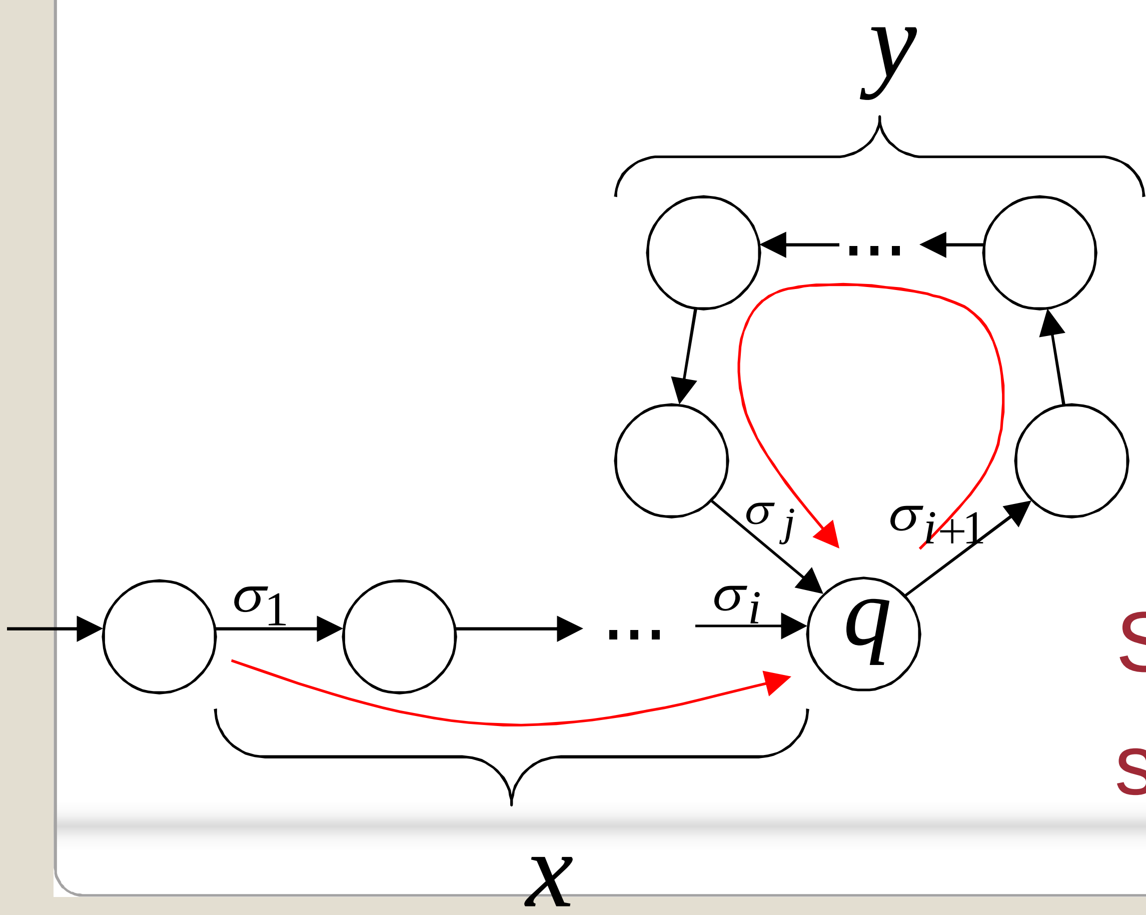


$x = \sigma_1 \cdots \sigma_i$ $\quad y = \sigma_{i+1} \cdots \sigma_j$ $\quad z = \sigma_{j+1} \cdots \sigma_k$

In DFA: $w = x\ y\ z$



Observation: length $|x\,y| \leq m$ number of states of DFA



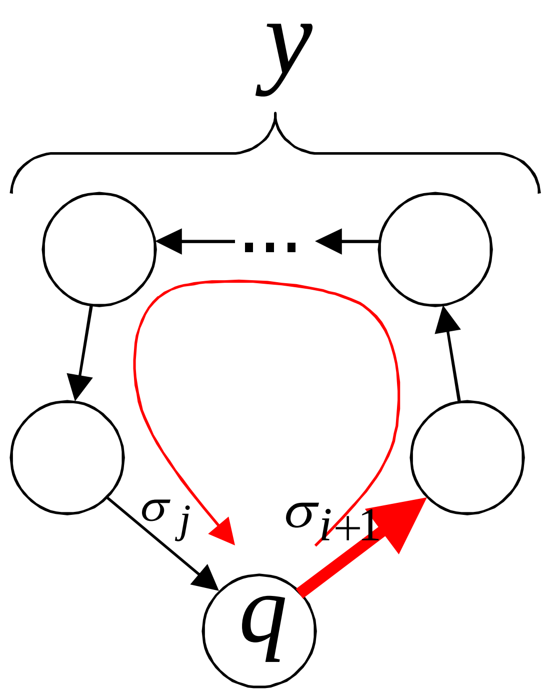
Unique States

Since, in xy no state is repeated

(except q)

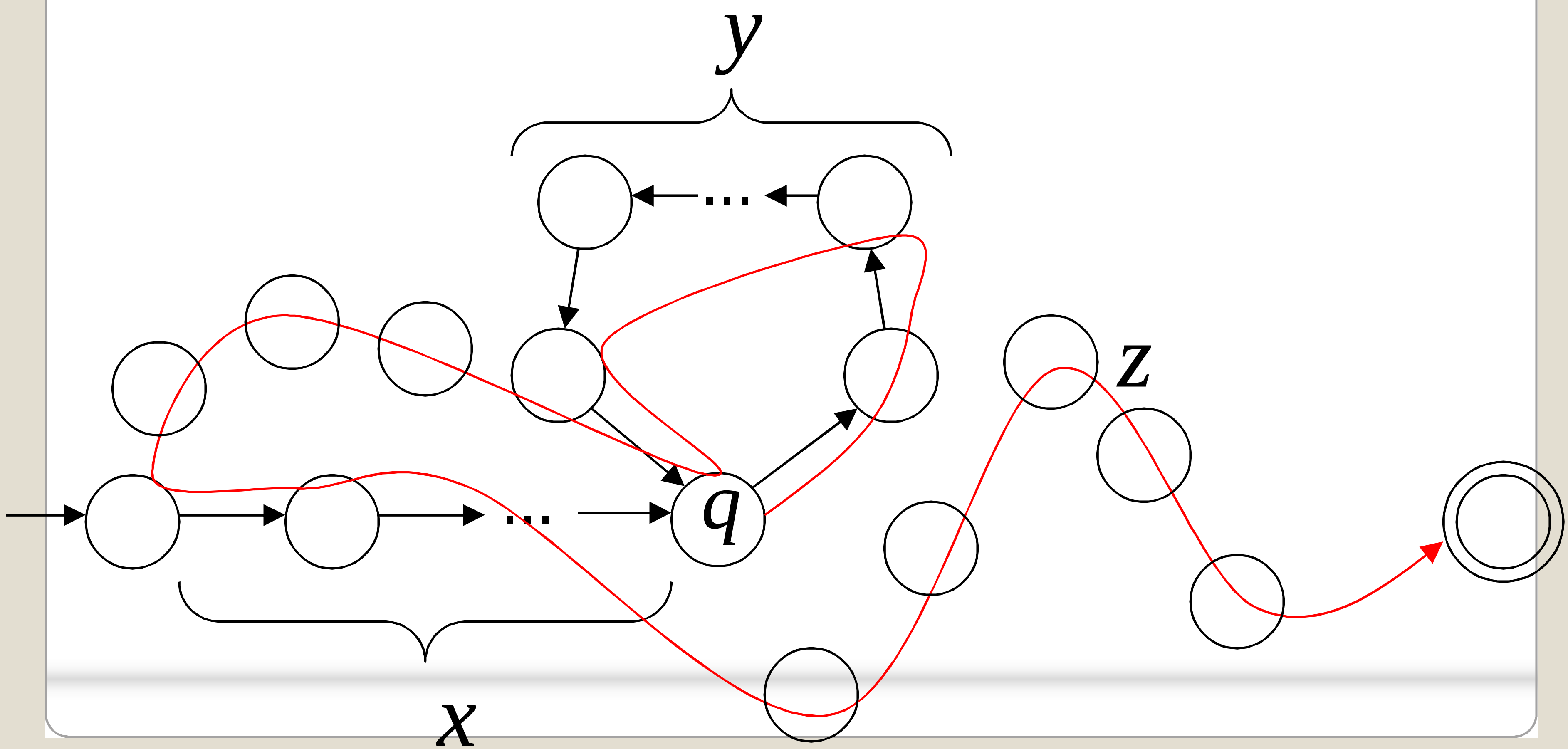
Observation: length $|y| \geq 1$

Since there is at least one transition in loop



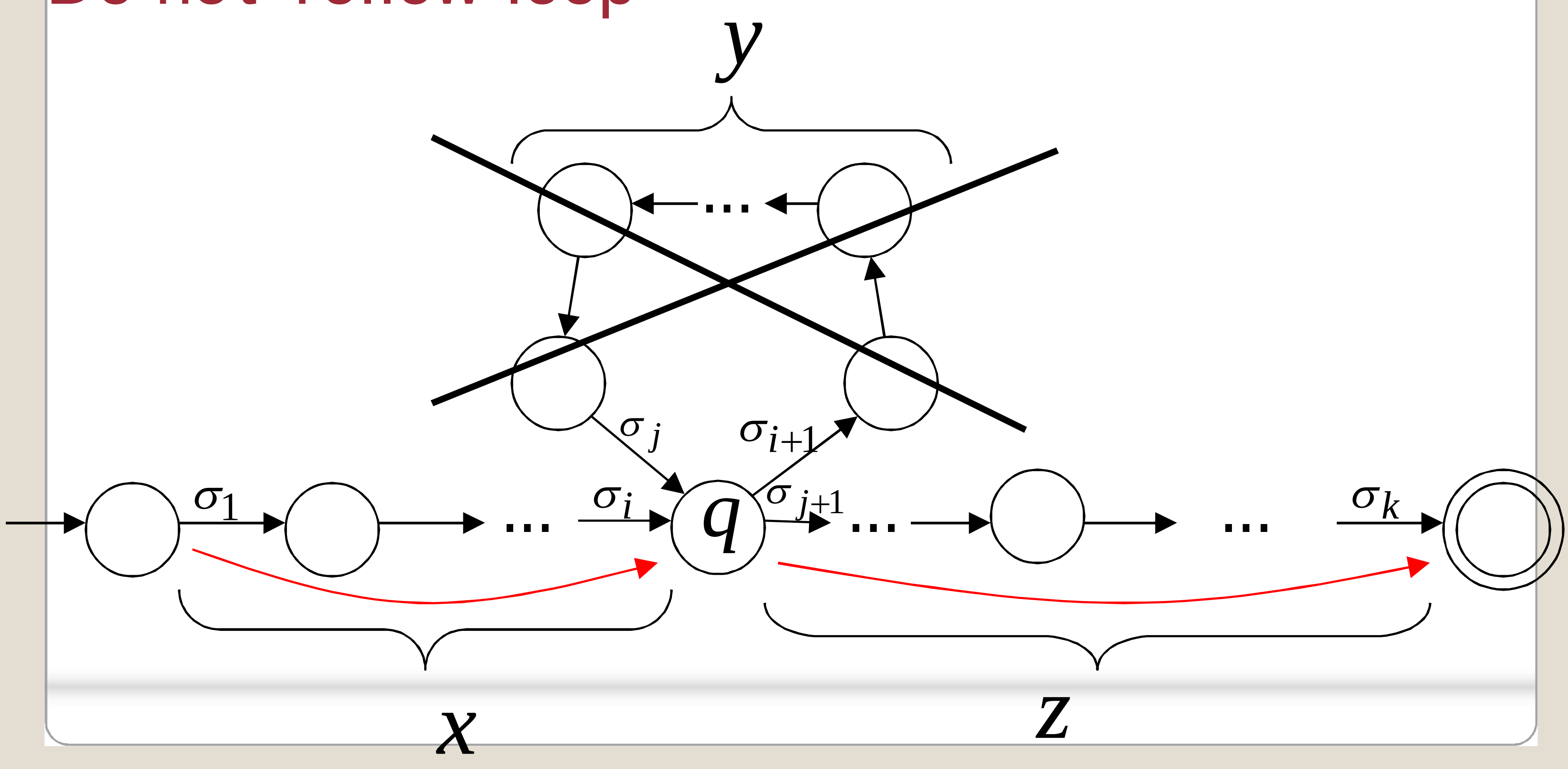
We do not care about the form of string z

z may actually overlap with the paths of x and y



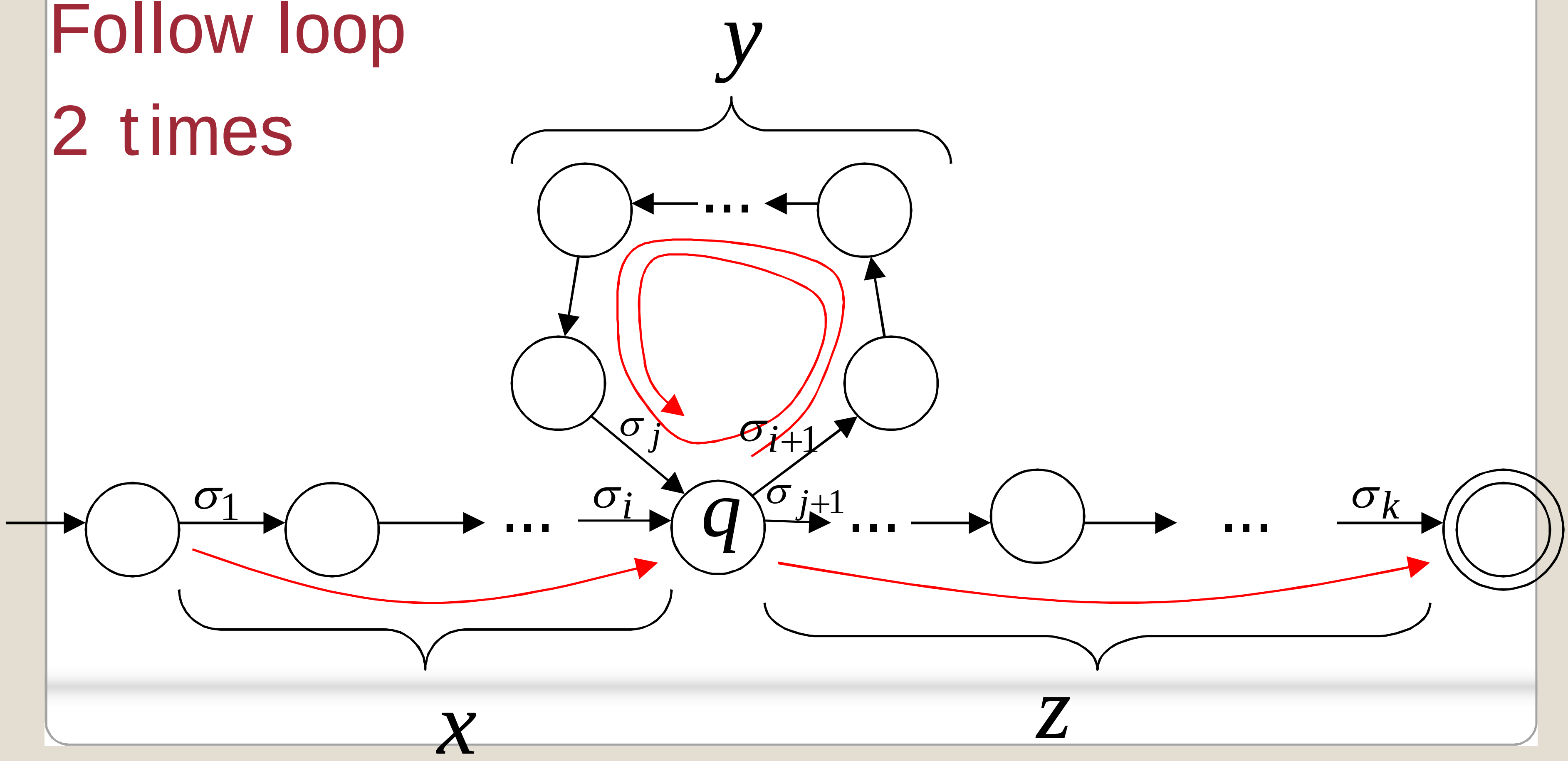
Additional string: The string $x\,z$ is accepted

Do not follow loop



Additional string: The string $x\ y\ y\ z$ is accepted

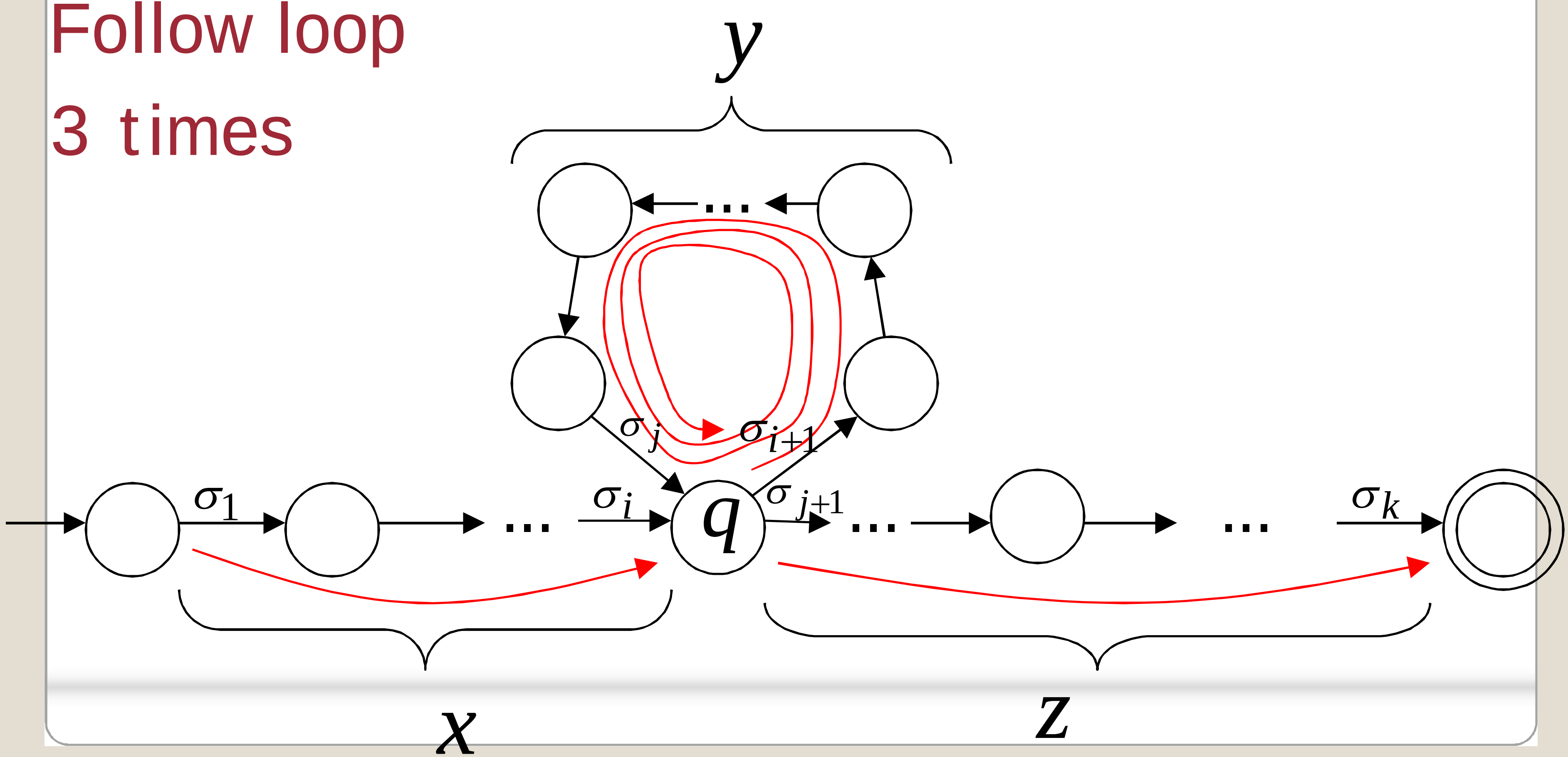
Follow loop 2 times



Additional string:

The string $x\ y\ y\ y\ z$ is accepted

Follow loop 3 times

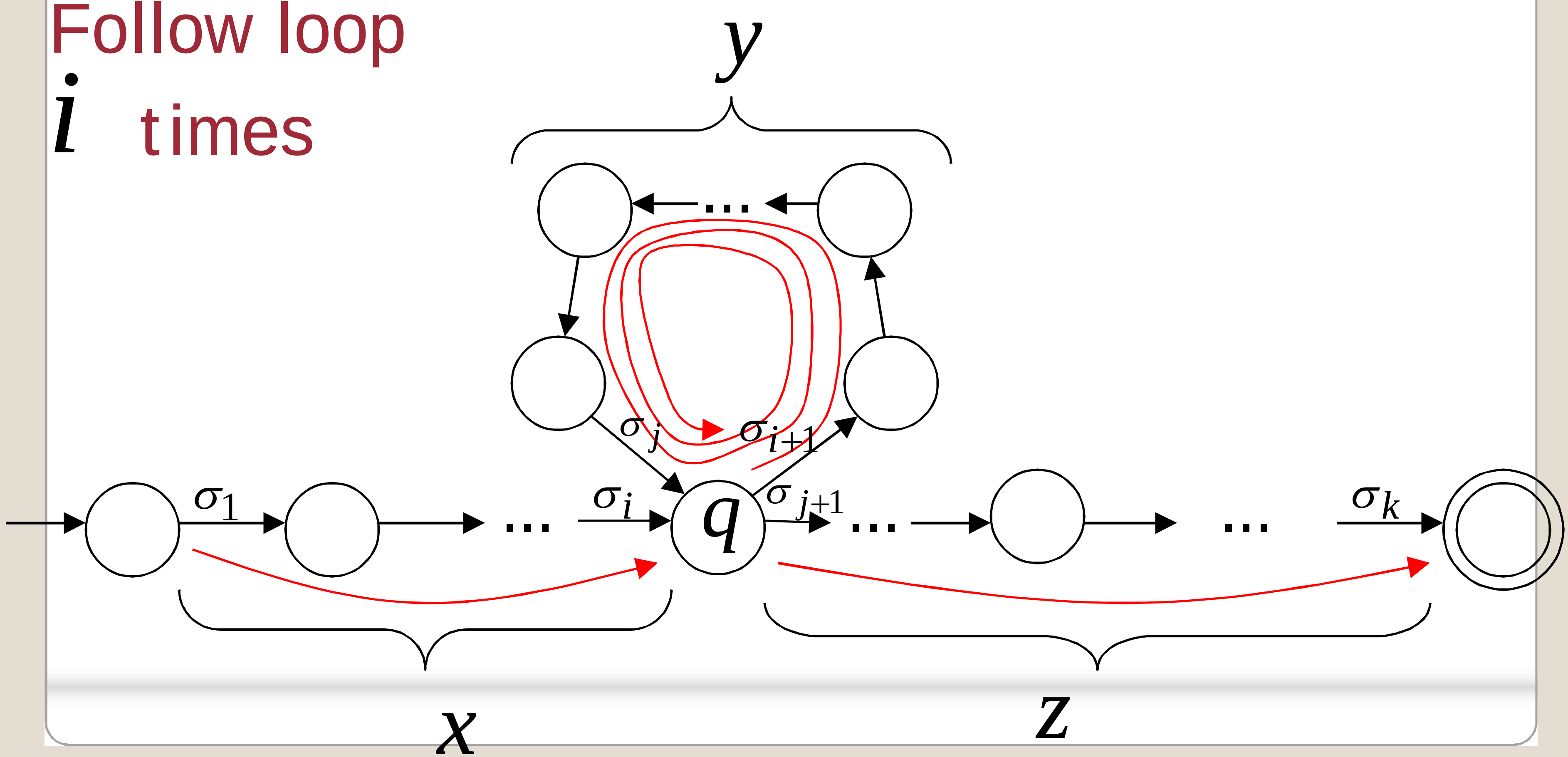


In General: The string $x\,y^{i}\,z$

is accepted $i = 0, 1, 2, \ldots$

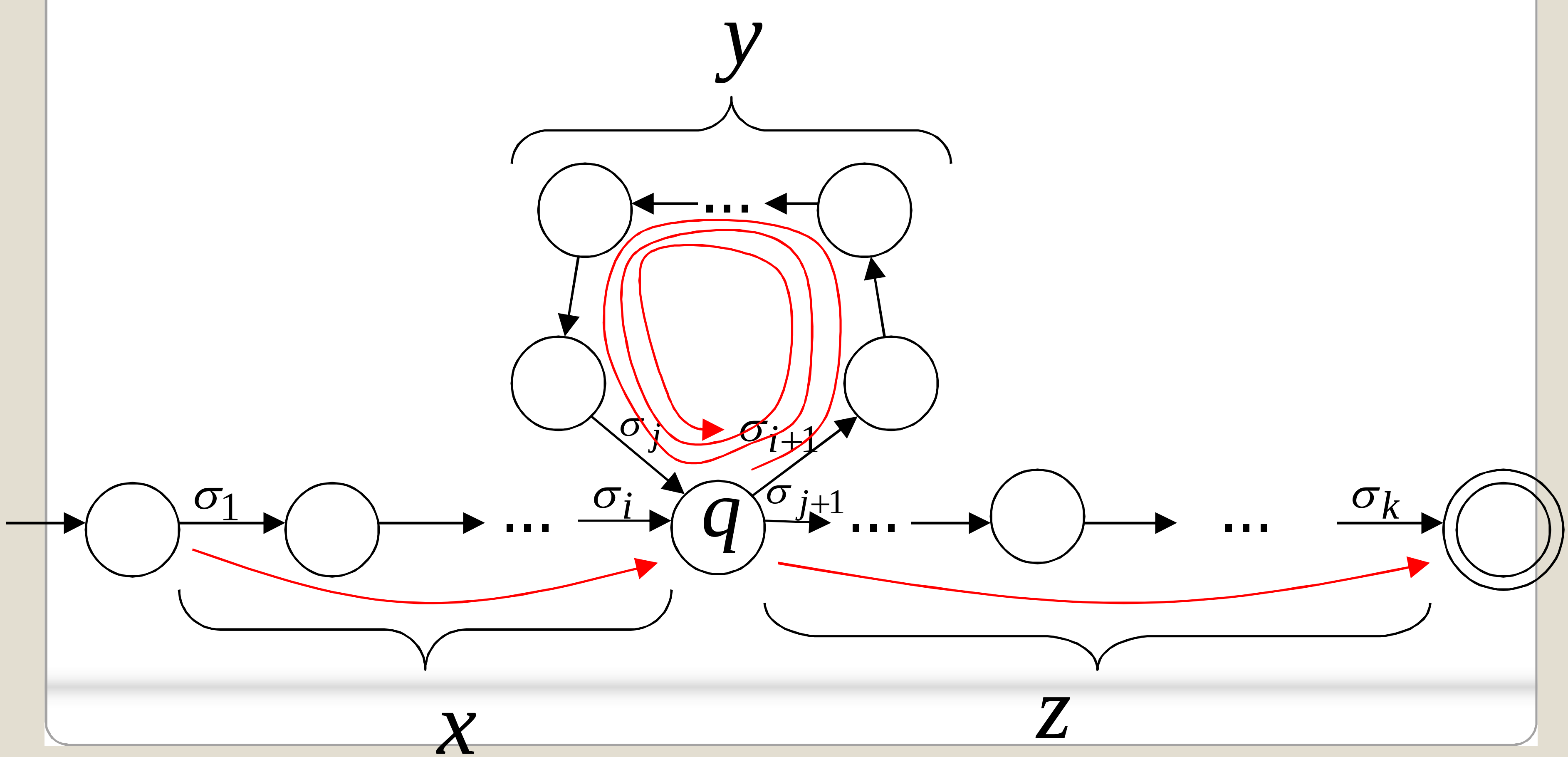
Follow loop

i times



Therefore: $x\, y^i\, z \in L \qquad i = 0, 1, 2, \ldots$

Language accepted by the DFA



In other words, we described:

The Pumping Lemma !!!

The Pumping Lemma:

- Given a infinite regular language L
- there exists an integer m (critical length)
- for any string $w \in L$ with length $|w| \geq m$
- we can write $w = x\,y\,z$
- with $|x\,y| \leq m$ and $|y| \geq 1$
- such that: $x\,y^i\,z \in L$ $\quad i = 0, 1, 2, \ldots$

In the book:

Critical length m = Pumping length p

Applications of the Pumping Lemma

Observation:

- Every language of finite size has to be regular

  (we can easily construct an NFA
  that accepts every string in the language)

Therefore, every non-regular language
has to be of infinite size

(contains an infinite number of strings)

Suppose you want to prove that
An infinite language L is not regular

1. Assume the opposite: L is regular

2. The pumping lemma should hold for L

3. Use the pumping lemma to obtain a contradiction

4. Therefore, L is not regular

Explanation of Step 3: How to get a contradiction

1. Let m be the critical length for L

2. Choose a particular string $w \in L$ which satisfies the length condition $|w| \geq m$

3. Write $w = xyz$

4. Show that $w' = xy^i z \notin L$ for some $i \neq 1$

5. This gives a contradiction, since from pumping lemma $w' = xy^i z \in L$

Note: It suffices to show that only one string $w \in L$ gives a contradiction

You don't need to obtain contradiction for every $w \in L$

Example of Pumping Lemma application

Theorem: The language $L = \{a^n b^n : n \geq 0\}$ is not regular

Proof: Use the Pumping Lemma

$$L = \{a^n b^n : n \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{a^n b^n : n \geq 0\}$$

Let m be the critical length for L

Pick a string w such that: $w \in L$

and length $|w| \geq m$

We pick $w = a^m b^m$

From the Pumping Lemma:

we can write $w = a^m b^m = x\ y\ z$

with lengths $|x\,y| \leq m, \quad |y| \geq 1$

$$w = xyz = a^m b^m = \overbrace{a...aa...aa...a}^{m}\overbrace{b...b}^{m}$$

$$\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...ab...b}_{z}$$

Thus: $y = a^k, \quad 1 \leq k \leq m$

$$x\,y\,z = a^m b^m \qquad y = a^k, \quad 1 \le k \le m$$

From the Pumping Lemma: $x\,y^i\,z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\,y^2\,z \in L$

$x\,y\,z = a^m b^m \qquad y = a^k, \quad 1 \le k \le m$

From the Pumping Lemma: $x\,y^2\,z \in L$

$$xy^2z = \overbrace{a...aa...aa...aa...a}^{m+k}\overbrace{b...b}^{m} \in L$$

$$xy^2z = \underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}\underbrace{a...ab...b}_{z} \in L$$

Thus: $a^{m+k}b^m \in L$

$$a^{m+k}b^m \in L \qquad k \geq 1$$

BUT: $L = \{a^n b^n : n \geq 0\}$

⬇

$$a^{m+k}b^m \notin L$$

CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

END OF PROOF

Non-regular language $\{a^n b^n : \ n \geq 0\}$